What Periodic Trajectories Exist for a Particle Subjected to a Central Force $F = -r^{-\frac{1}{2}}$?

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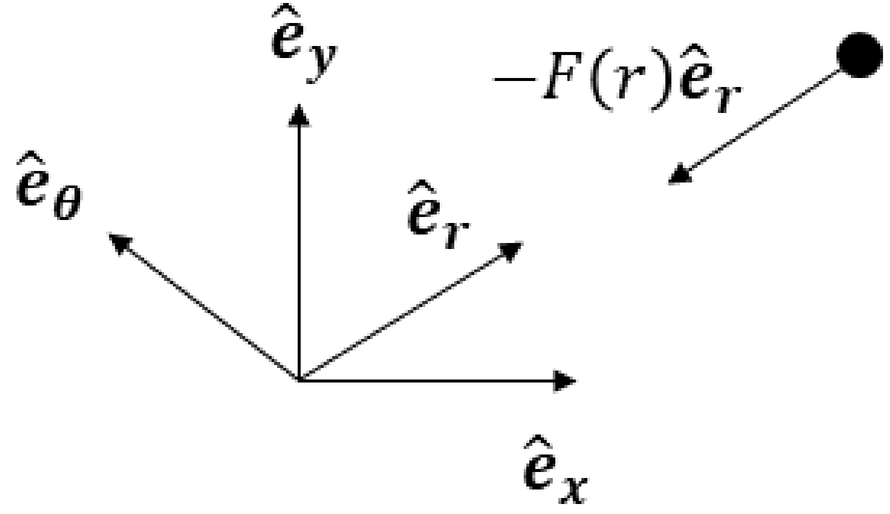
Background

- A central force is one that can be that be written as $\boldsymbol{F} = F(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}}$.
- For the central force $F(r) = -r^{-\frac{1}{2}}$, the equations of motion in Cartesian are:

$$\ddot{x} = F\frac{x}{r}$$

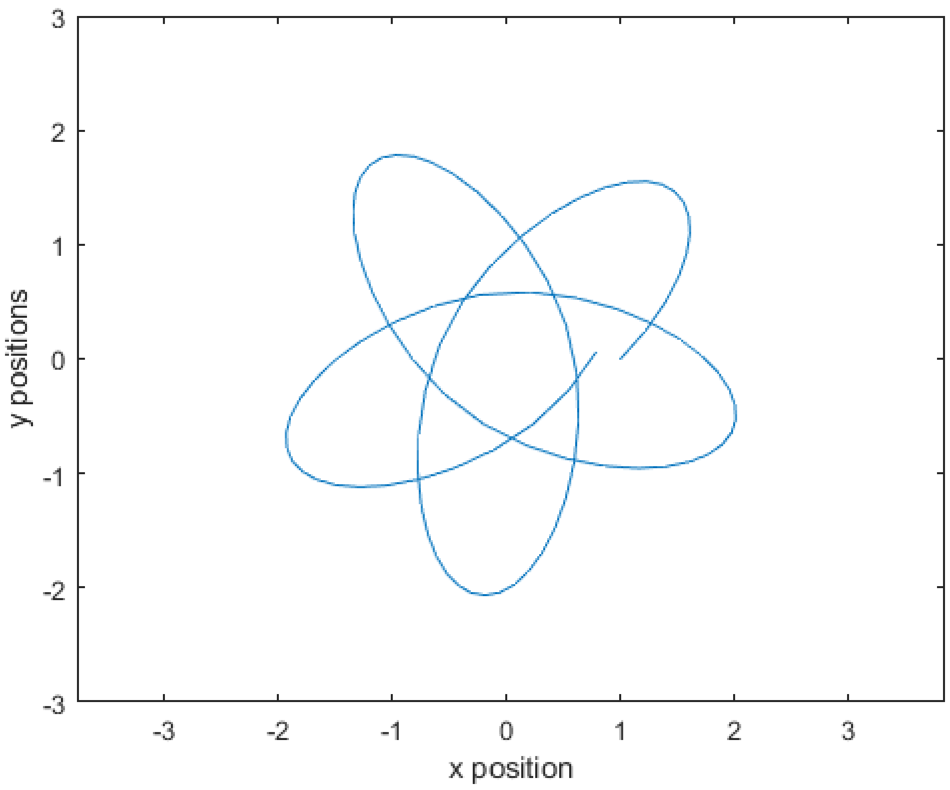
$$\ddot{y} = F\frac{y}{r}$$

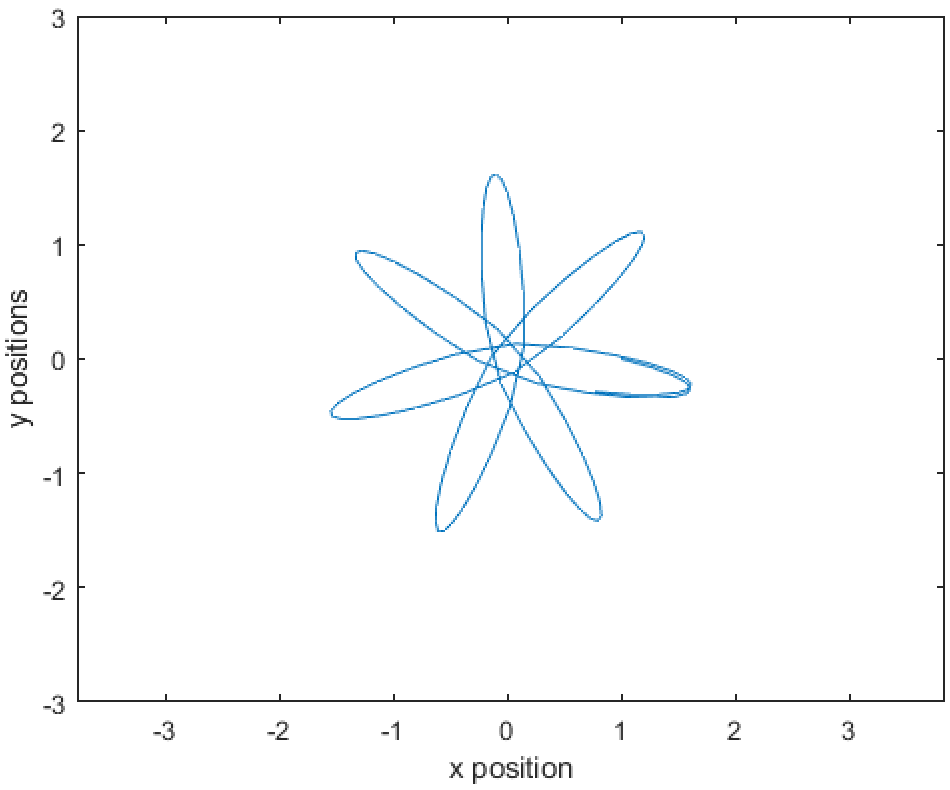
$$r = \sqrt{x^2 + y^2}$$



Motivation

There appears to exist periodic trajectories that are neither linear nor circular when subjected to the central force $F(r) = r^{-\frac{1}{2}}$.





Finding Possible Solutions

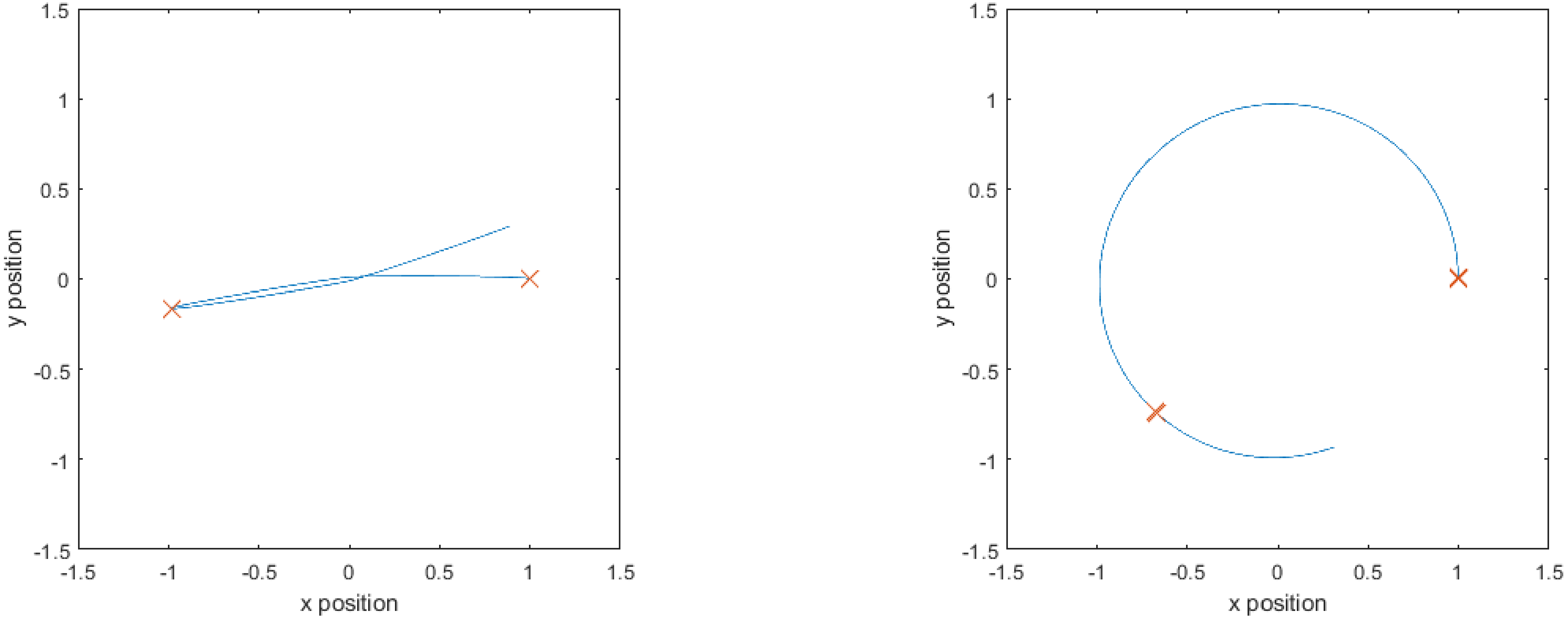
- All trajectories lie between a line and a circle.
  - Linear trajectory has no velocity or acceleration in the $\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$ direction.
  - Circular trajectory has no acceleration in the $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$ direction.
- How fast does a particle need to be traveling to move in a circle?
  - Written in polar, the acceleration after setting $\dot{r} = \ddot{r} = 0$ is

$$\boldsymbol{a} = -r\dot{\theta}^2\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\ddot{\theta}\hat{\boldsymbol{e}}_{\boldsymbol{\theta}} = \frac{\boldsymbol{v}^2}{\boldsymbol{r}}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\ddot{\theta}\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$$

  - Use LMB and dot in $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$: $v = r^{\frac{1}{4}}$
- Start particle with $[x\ y\ \dot{x}\ \dot{y}]$ = $[1\ 0\ 0\ v]$, $v\ \in [0, 1]$.
- Look at extremes of the velocity, specifically when v is close to 0 and when v *is close to 1.*

Boundaries of Possible Trajectories

- The location of the next lobe is bounded by ~(180°, 227.7°) away from the first lobe.
- This corresponds with travelling ~(0.5, 0.6325) of a circle.



Possible Solutions

Green is a possible lobed solution, yellow is the linear trajectory, orange is a repeated solution (and therefore not possible).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0.5	0.333333	0.25	0.2	0.166667	0.142857	0.125	0.111111	0.1	0.090909	0.083333	0.076923	0.071429	0.066667	0.0625	0.058824	0.055556	0.052632	0.05
2		1	0.666667	0.5	0.4	0.333333	0.285714	0.25	0.222222	0.2	0.181818	0.166667	0.153846	0.142857	0.133333	0.125	0.117647	0.111111	0.105263	0.1
3			1	0.75	0.6	0.5	0.428571	0.375	0.333333	0.3	0.272727	0.25	0.230769	0.214286	0.2	0.1875	0.176471	0.166667	0.157895	0.15
4				1	0.8	0.666667	0.571429	0.5	0.444444	0.4	0.363636	0.333333	0.307692	0.285714	0.266667	0.25	0.235294	0.222222	0.210526	0.2
5					1	0.833333	0.714286	0.625	0.555556	0.5	0.454545	0.416667	0.384615	0.357143	0.333333	0.3125	0.294118	0.277778	0.263158	0.25
6						1	0.857143	0.75	0.666667	0.6	0.545455	0.5	0.461538	0.428571	0.4	0.375	0.352941	0.333333	0.315789	0.3
7							1	0.875	0.777778	0.7	0.636364	0.583333	0.538462	0.5	0.466667	0.4375	0.411765	0.388889	0.368421	0.35
8								1	0.888889	0.8	0.727273	0.666667	0.615385	0.571429	0.533333	0.5	0.470588	0.444444	0.421053	0.4
9									1	0.9	0.818182	0.75	0.692308	0.642857	0.6	0.5625	0.529412	0.5	0.473684	0.45
10										1	0.909091	0.833333	0.769231	0.714286	0.666667	0.625	0.588235	0.555556	0.526316	0.5
11											1	0.916667	0.846154	0.785714	0.733333	0.6875	0.647059	0.611111	0.578947	0.55
12												1	0.923077	0.857143	0.8	0.75	0.705882	0.666667	0.631579	0.6
13													1	0.928571	0.866667	0.8125	0.764706	0.722222	0.684211	0.65
14														1	0.933333	0.875	0.823529	0.777778	0.736842	0.7
15															1	0.9375	0.882353	0.833333	0.789474	0.75
16																1	0.941176	0.888889	0.842105	0.8
17																	1	0.944444	0.894737	0.85
18																		1	0.947368	0.9
19																			1	0.95
20																				1

No Lobed Solutions

- No solution containing 3 lobes, 4 lobes, 6 lobes, 10 lobes and 14 lobes.
- Check up to 3000, check in the same way to see if it’s between the two bounds.
- Appears that the only ones without solutions are the ones listed above.

Solutions

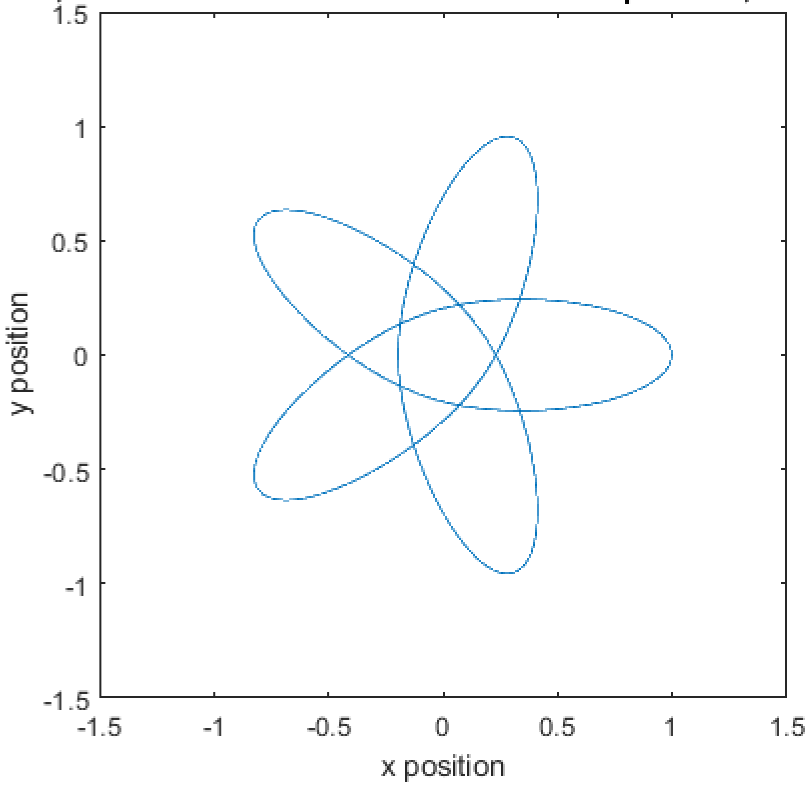
- Trajectory optimization was used to find periodic trajectories that are neither linear nor circular.
- The following solutions were primarily achieved using single shooting.
- The time it took the NLP to find a solution with the desired tolerance depended on:
  - Initial guess
  - Constraints
  - Number of parameters it could control

Helping the Trajectory Optimizer Converge

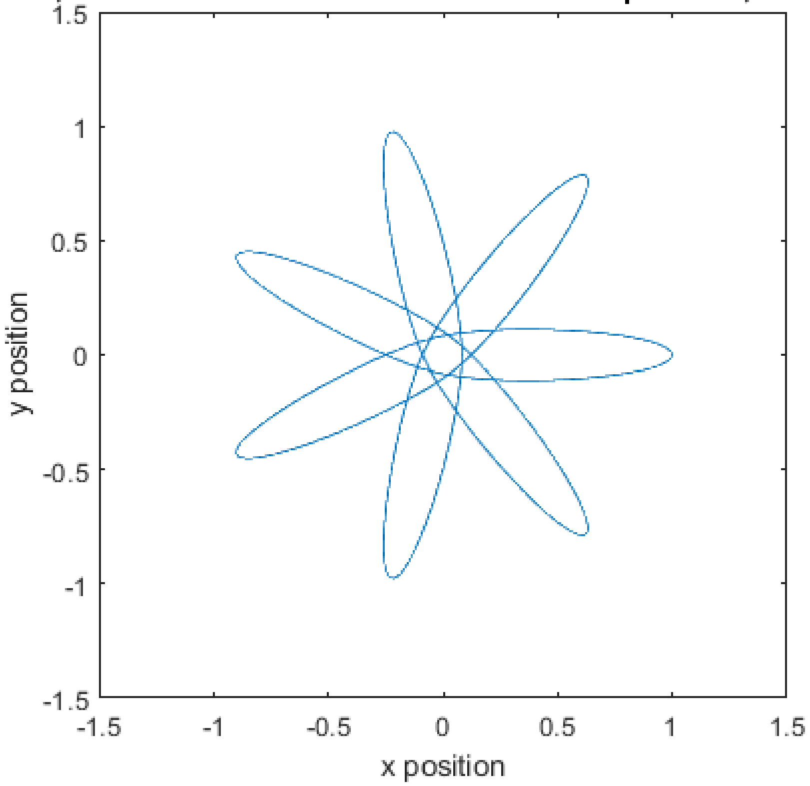
- Iterate. Start with a less tight tolerance, then step up the tolerance. Also start with a smaller number of points, and increase the number of points
- Impose constraints as tight as possible; double check that they are reasonable for the desired goal.
- Put at minimum a lower bound on time to prevent solutions that barely move.

Central Force $F = -r^{-\frac{1}{2}}$

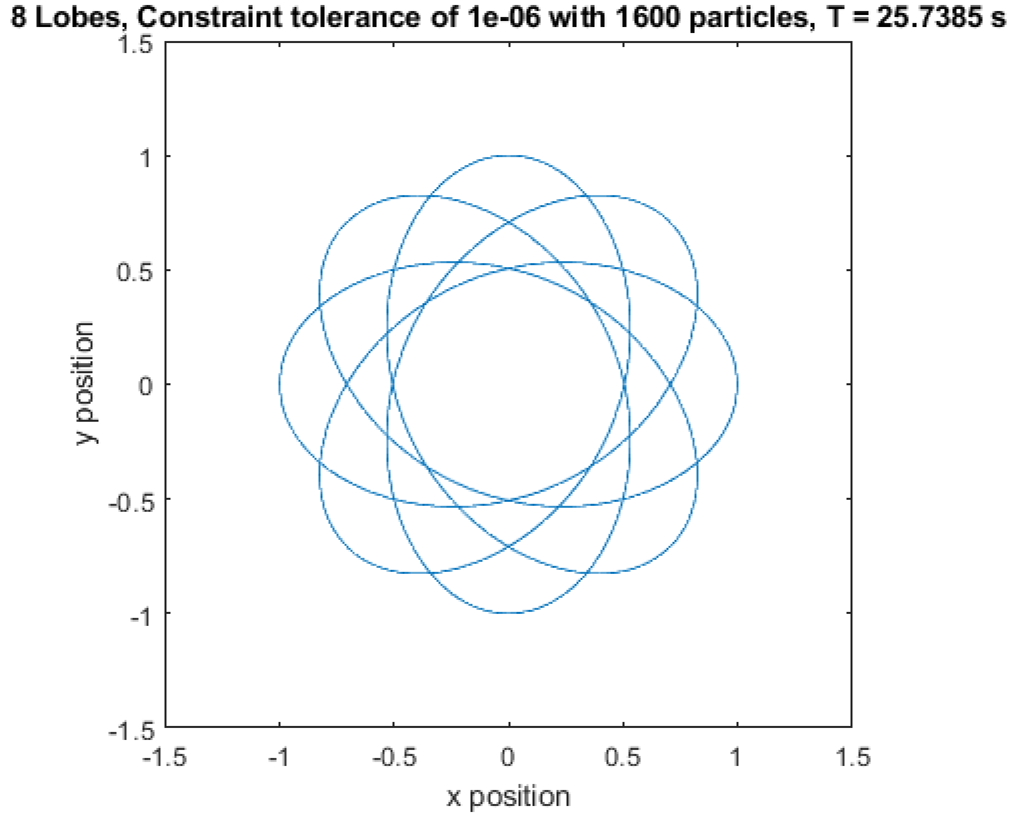
5 Lobes, Constraint tolerance of 1e-06 with 1600 particles, T = 14.1586 s

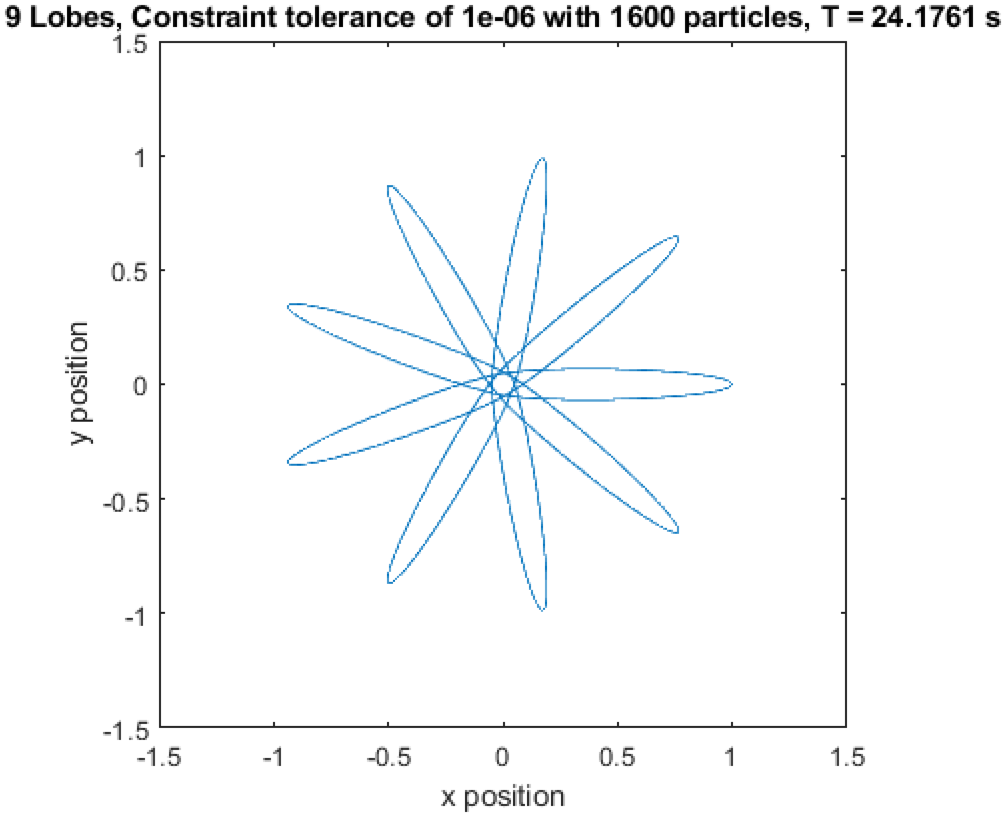


7 Lobes, Constraint tolerance of 1e-06 with 1600 particles, T = 18.9964 s



Central Force $F = -r^{-\frac{1}{2}}$





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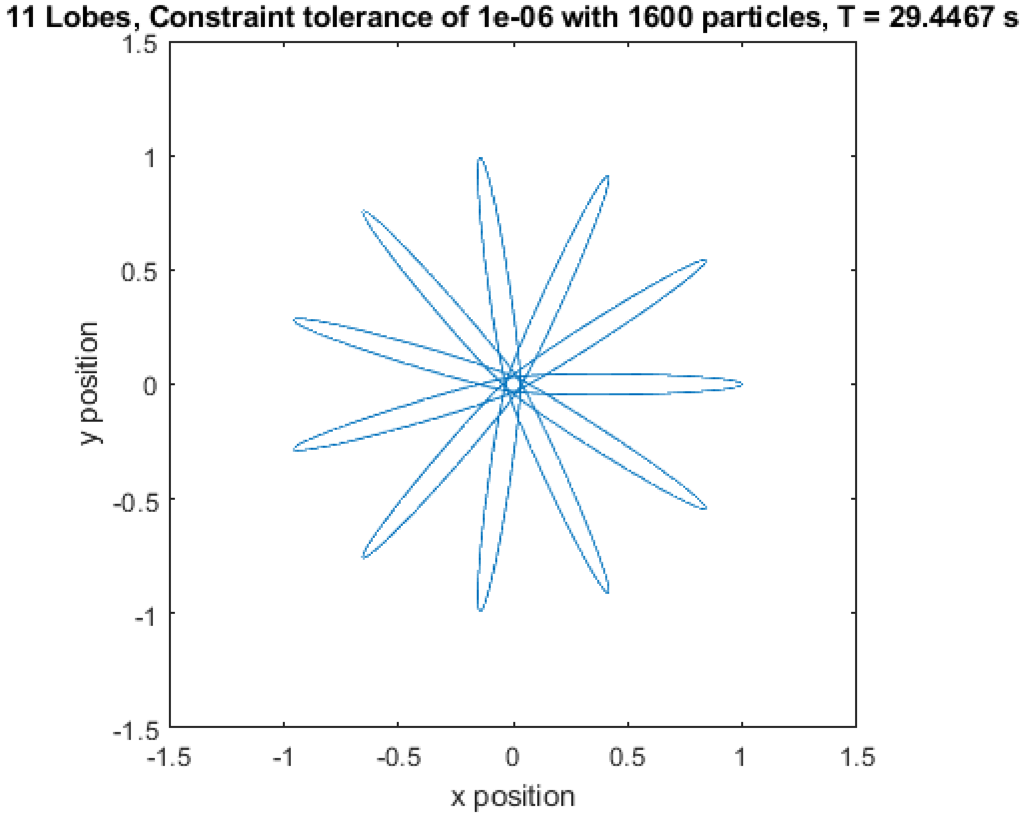

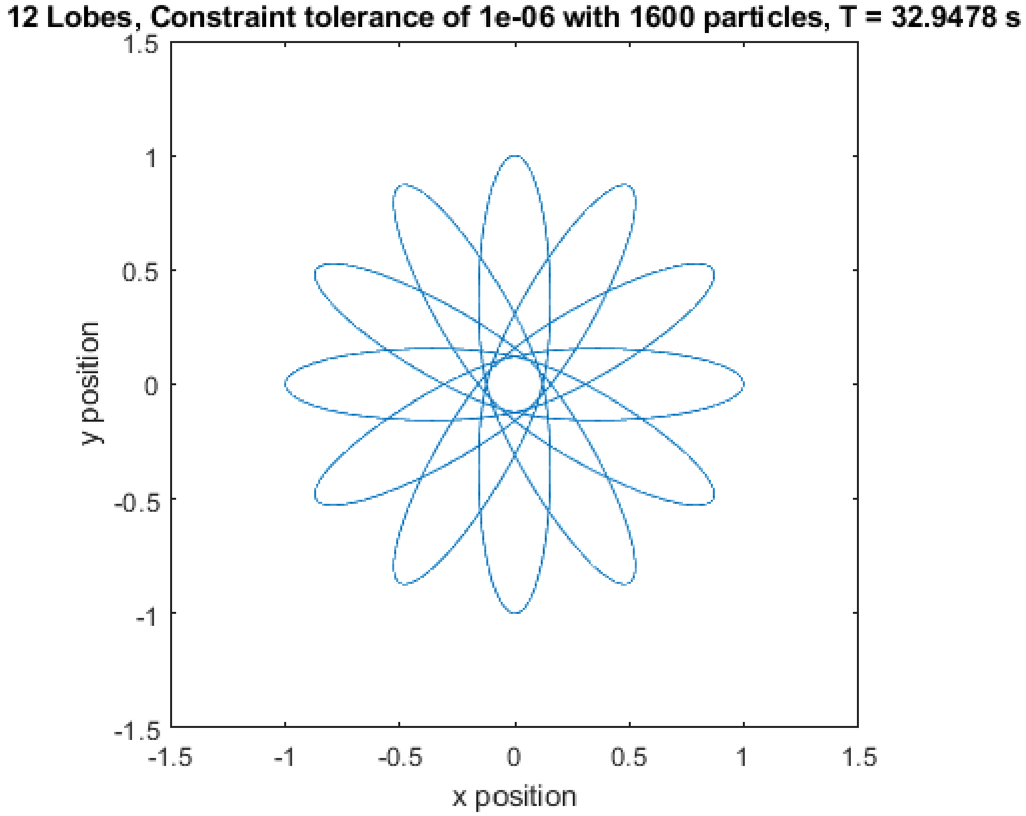

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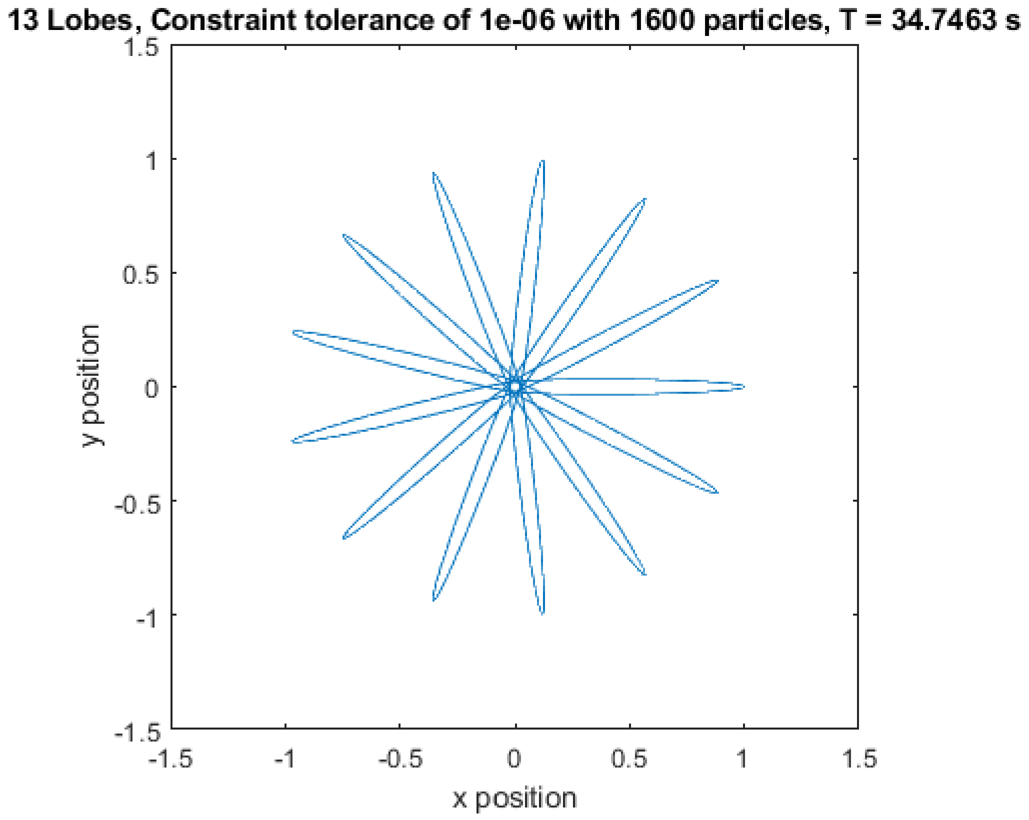

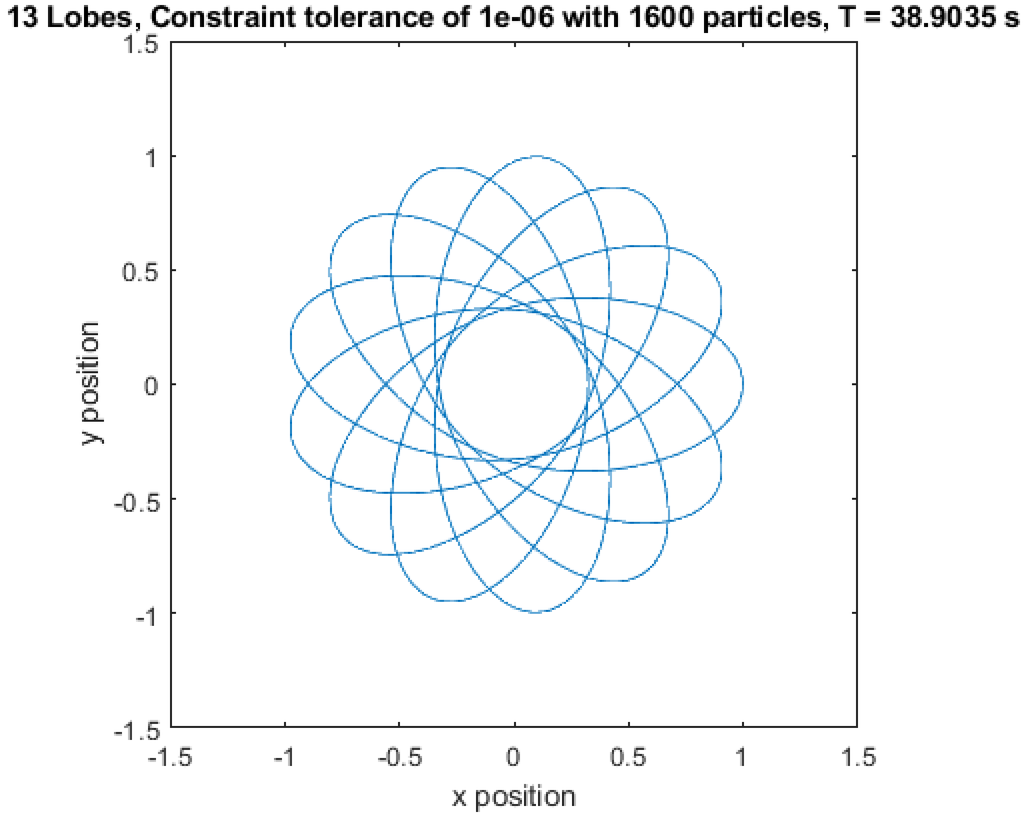

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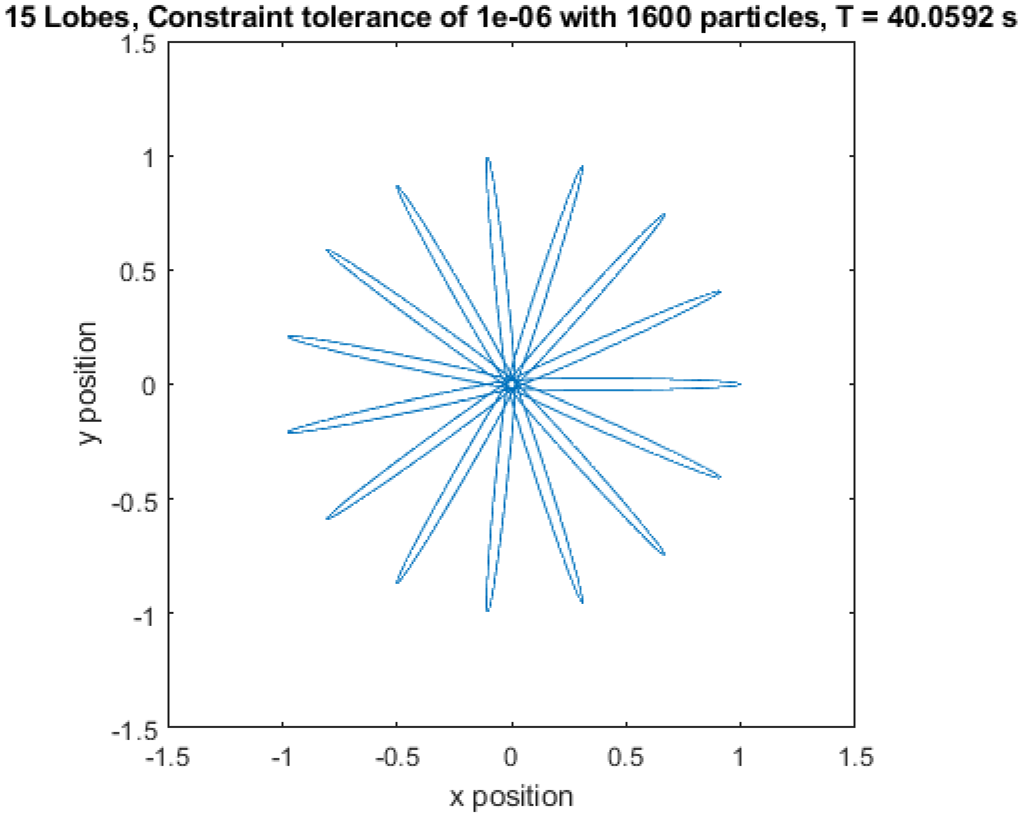

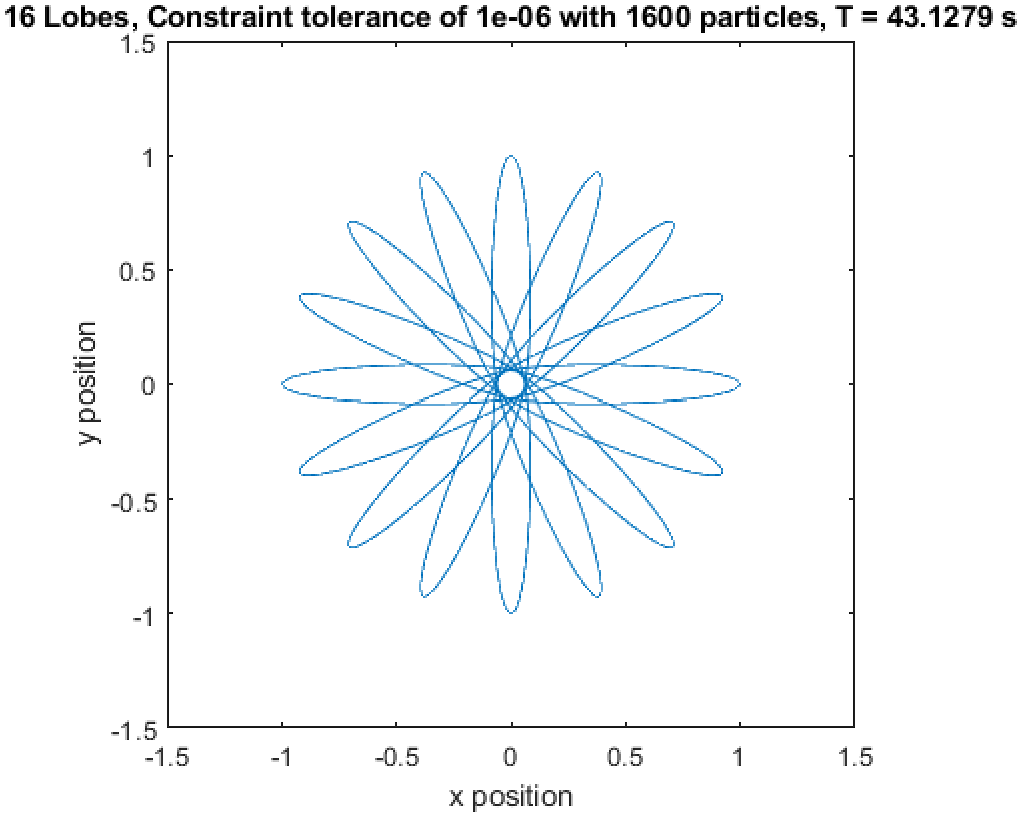

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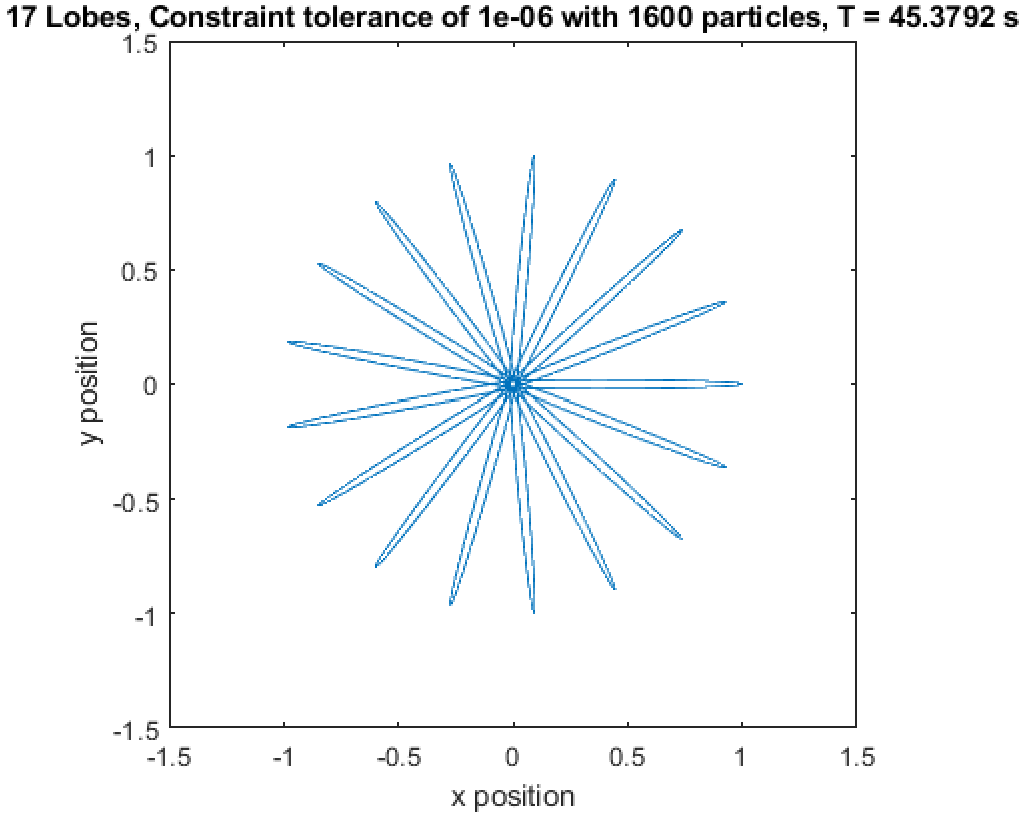

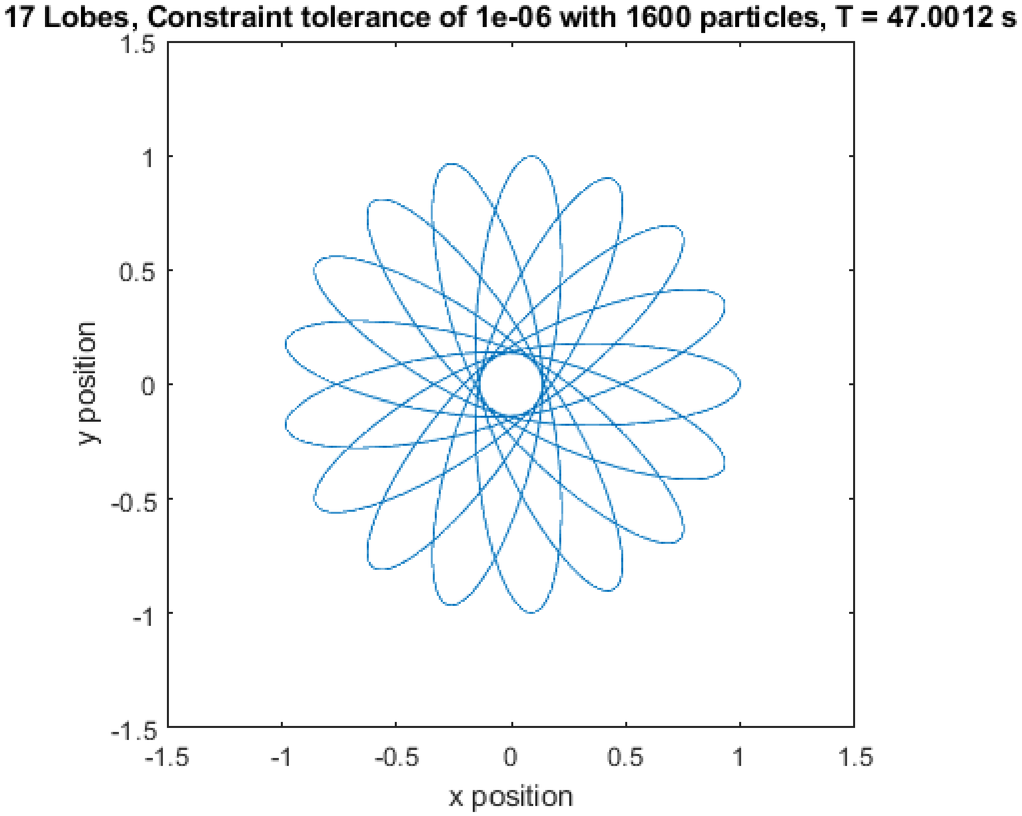

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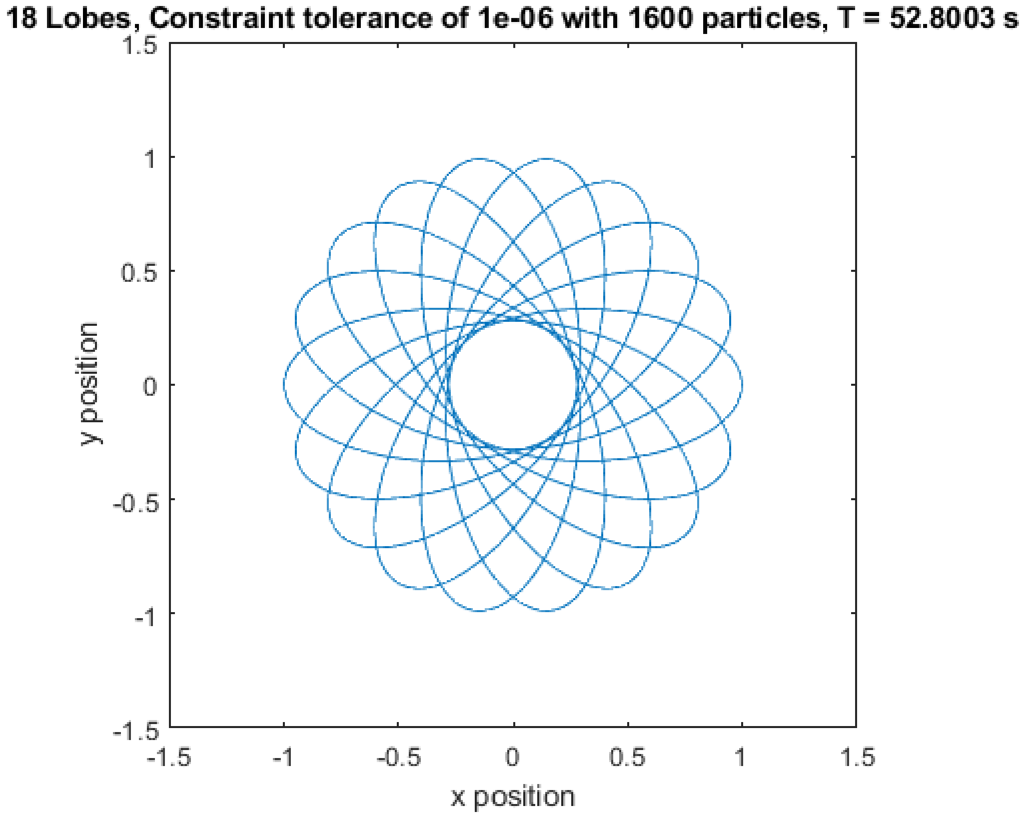

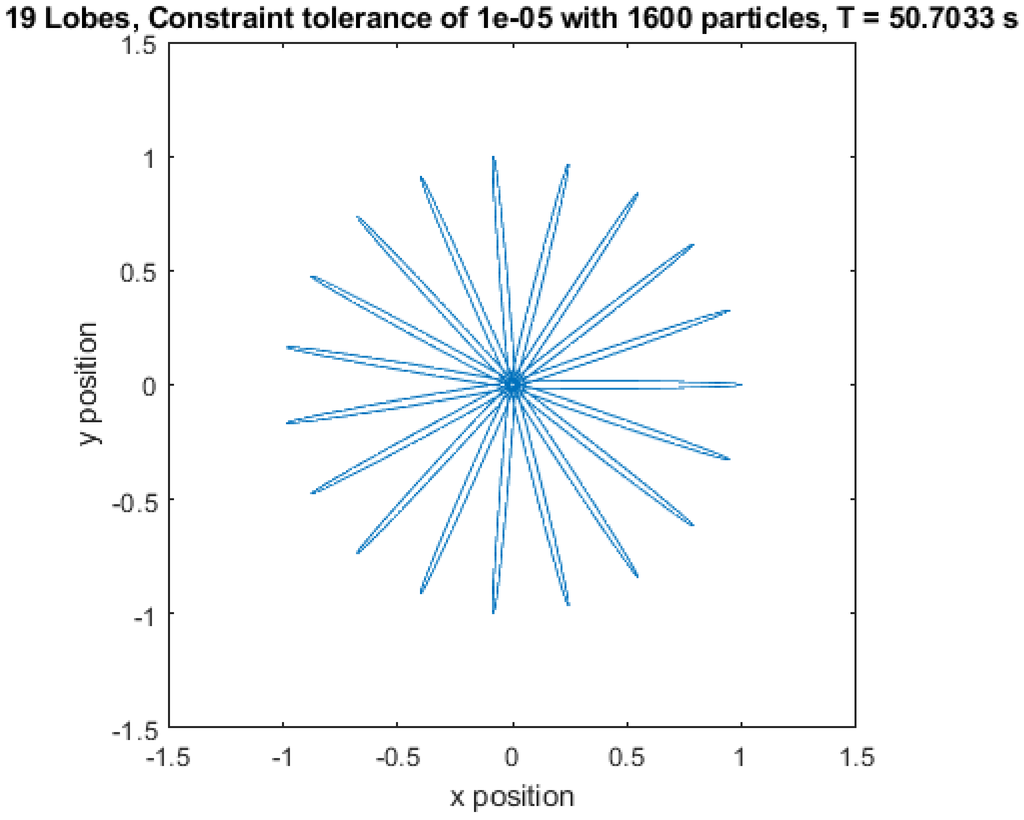

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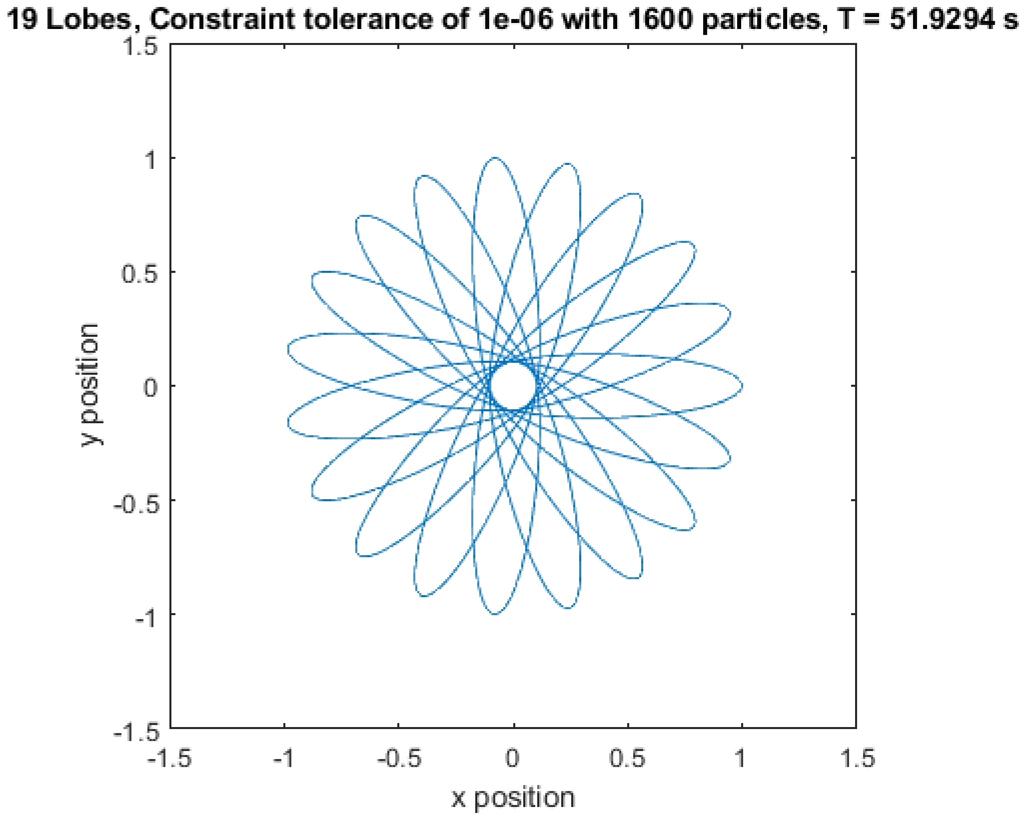

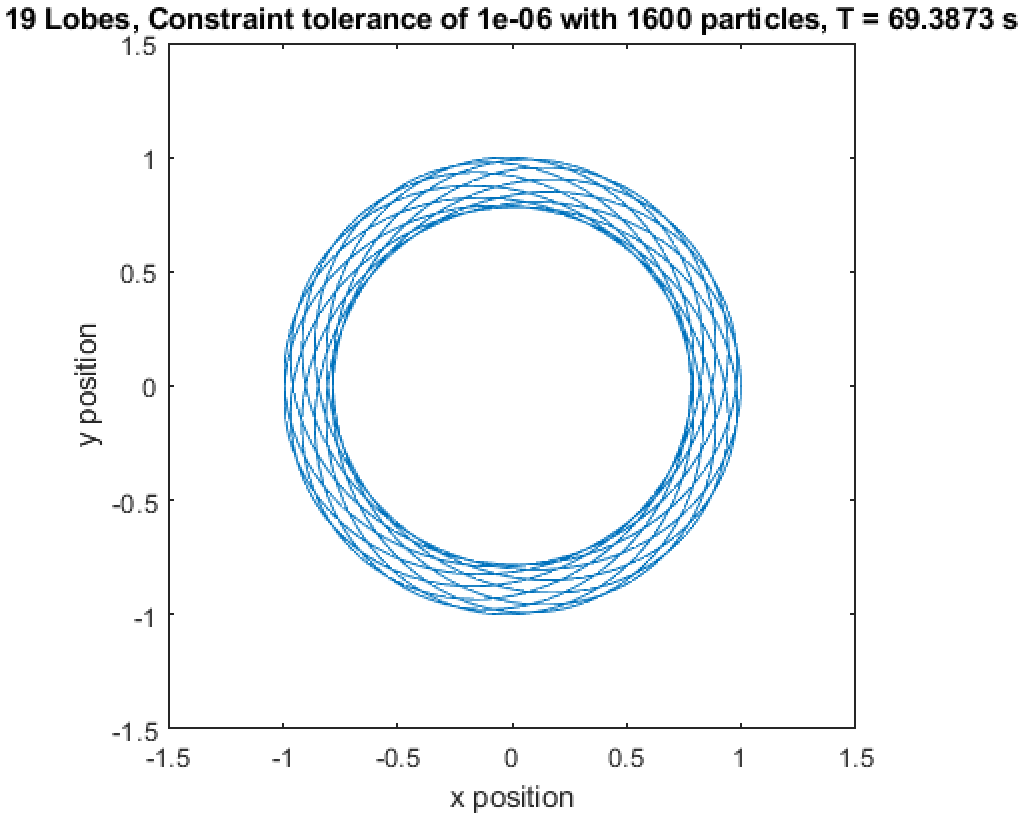

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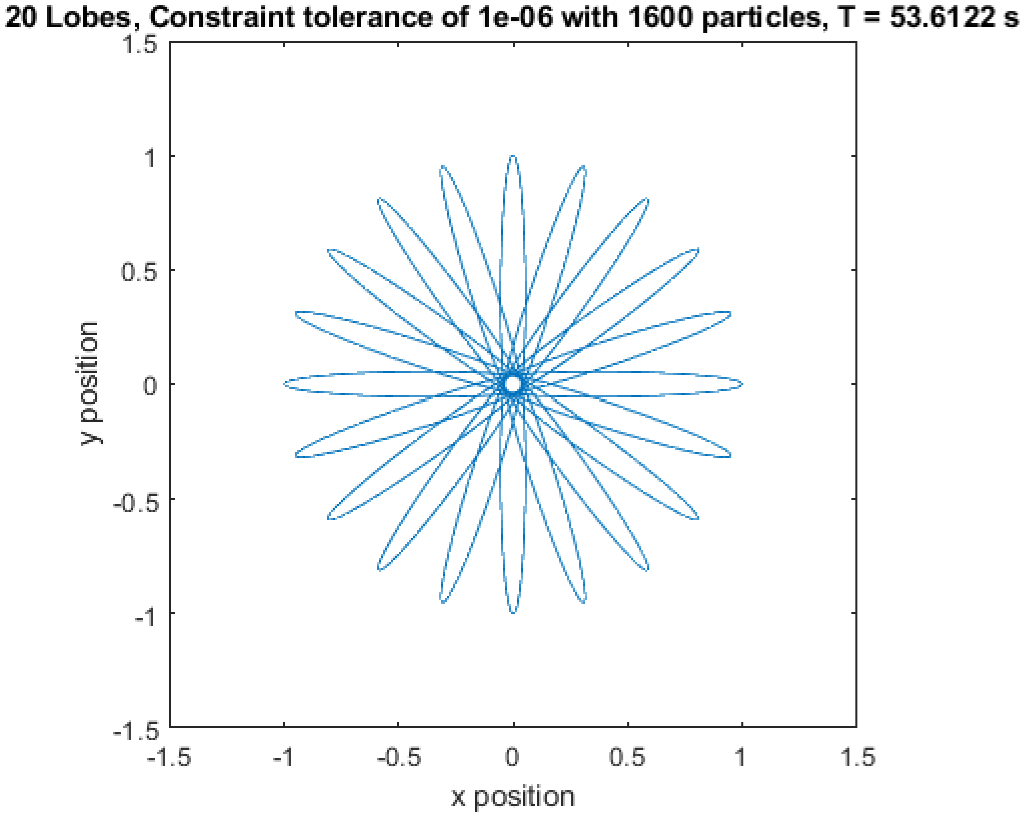
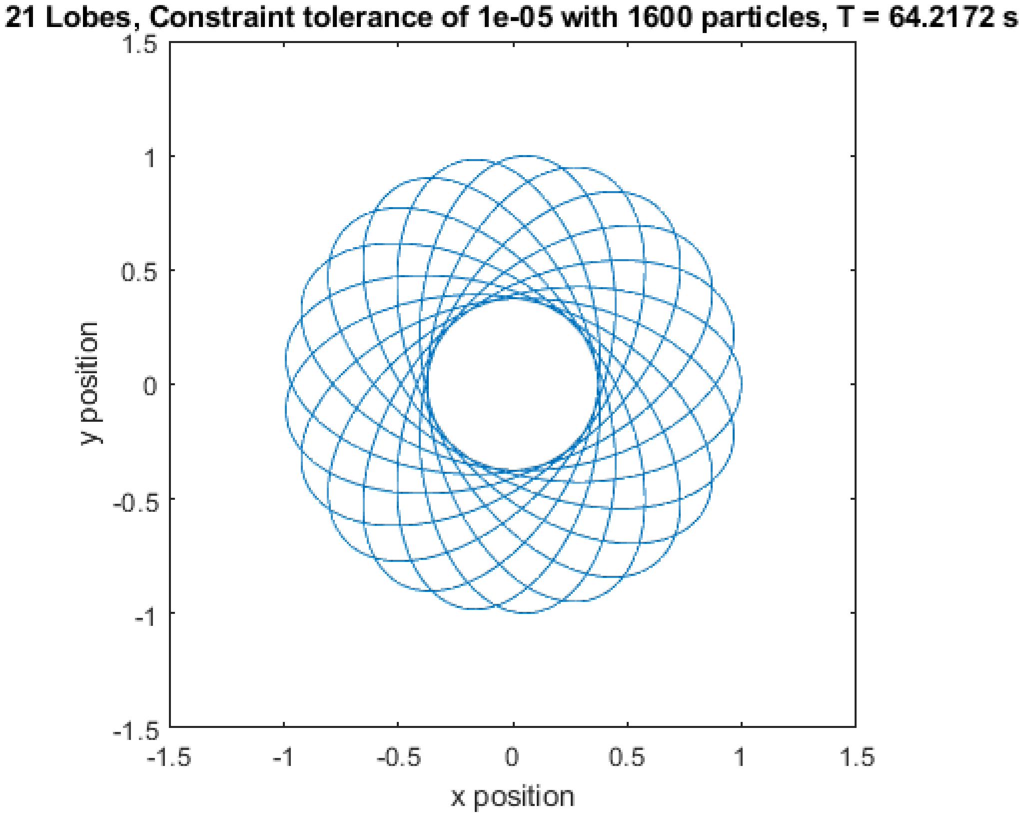


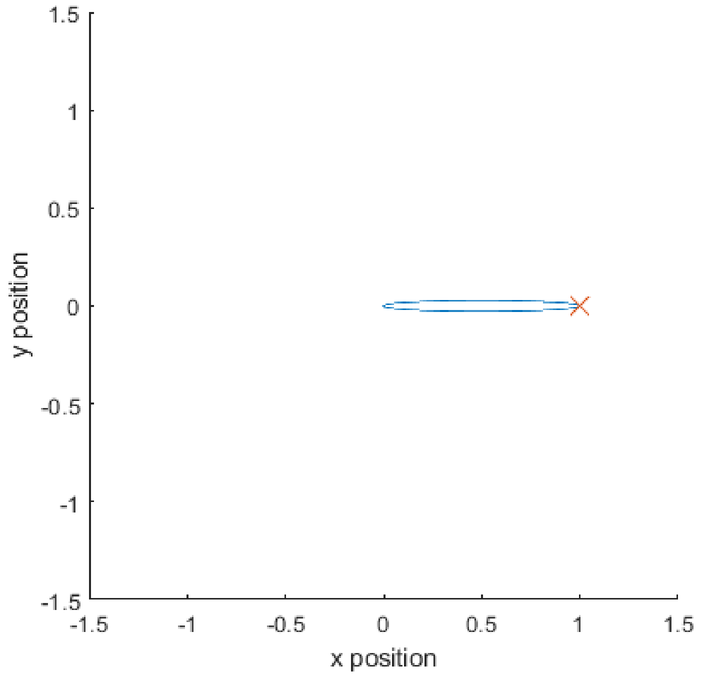


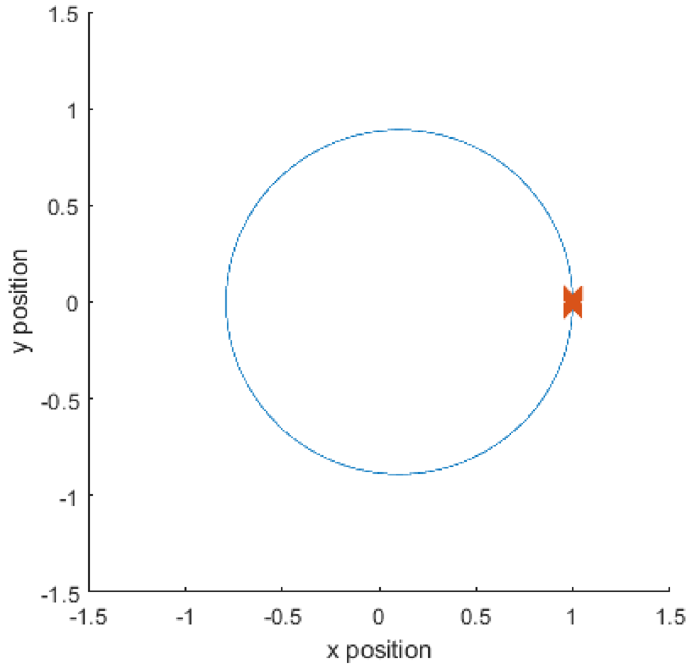
Single Shooting vs Multiple Shooting vs Collocation

- Attempt to converge to the same tolerance as that of ode45, so 1e-8.
- Multiple shooting takes a very long time to converge. It will do it, but very slowly.
- Collocation sometimes works better than single shooting, sometimes not.
- Multiple shooting and collocation work better if the initial guess that is not as good.

Other Central Forces

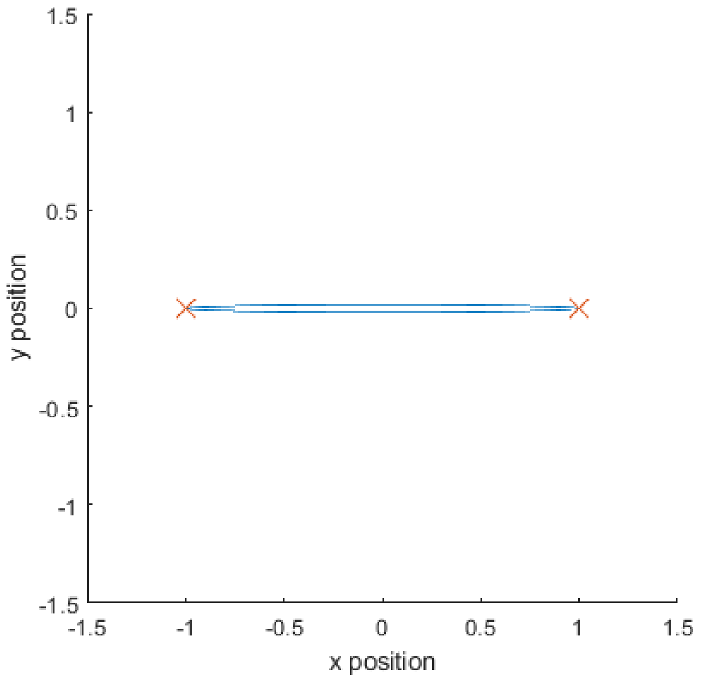
- $F = -r^{-2}$
- All trajectories are ellipses, or linear.
- The maximum must occur at the same point

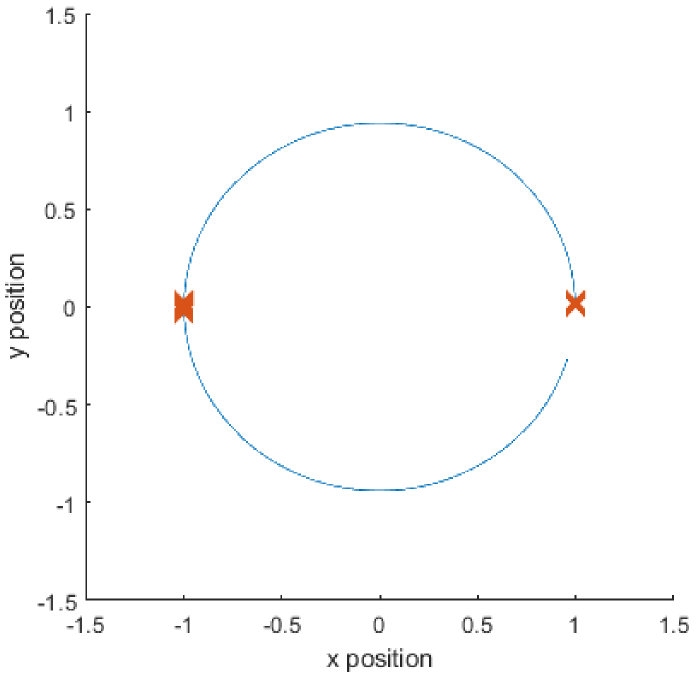




Other Central Forces

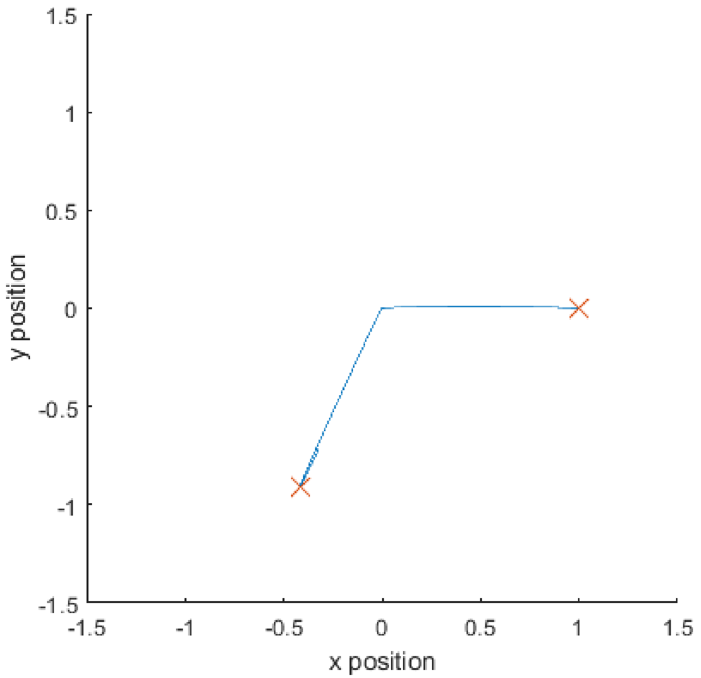
- $F = -r$
- All trajectories are ellipses, or linear.
- The maximum occurs 180º away.

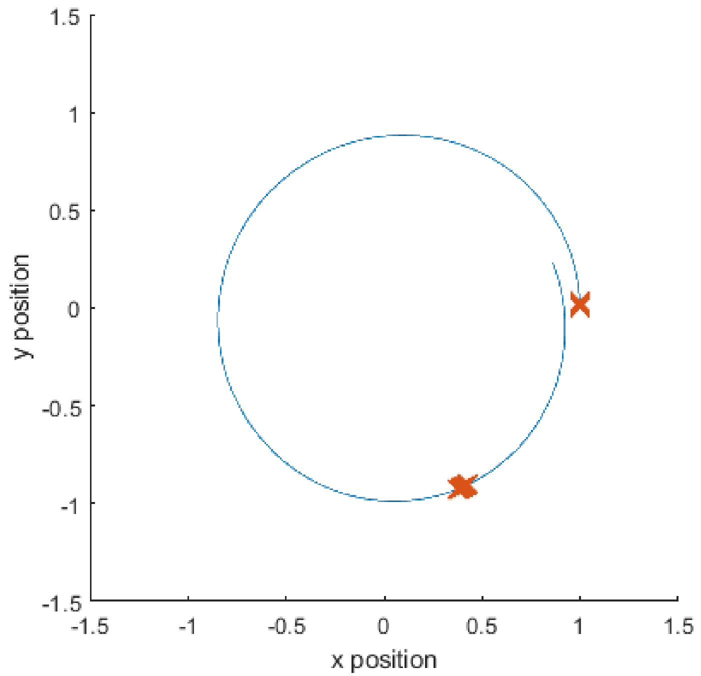




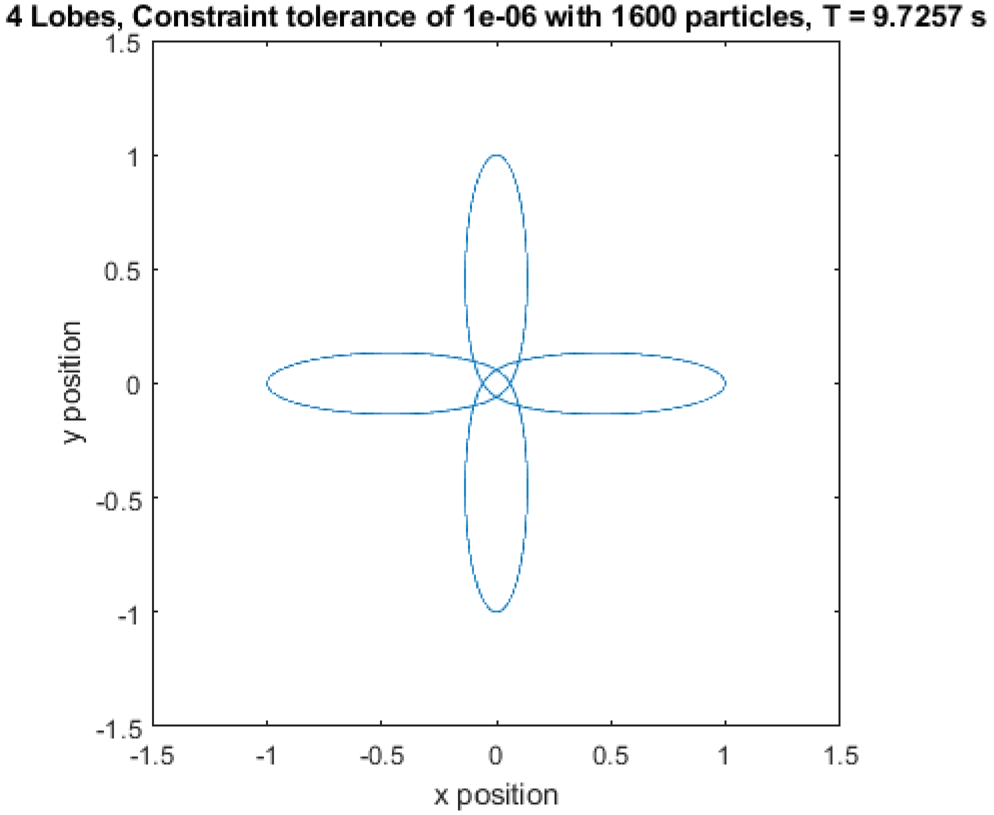
Other Central Forces

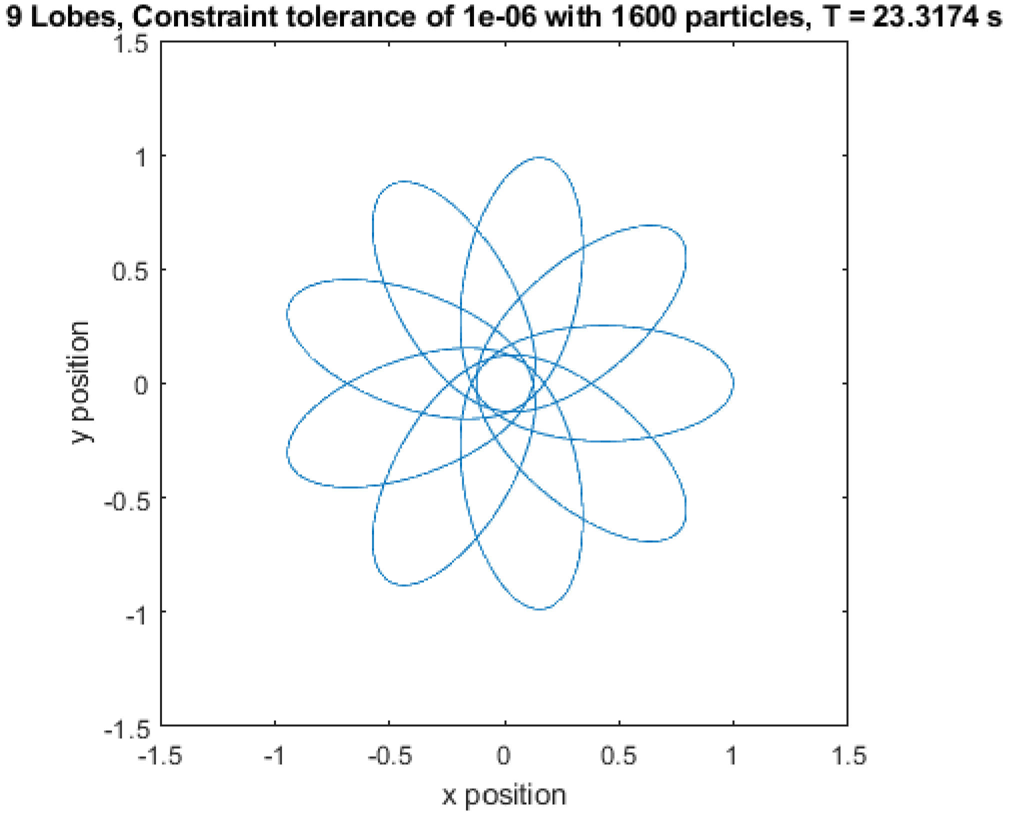
- $F = -r^{-\frac{3}{2}}$
- A different set of trajectories are possible/not possible.
- The location of the next lobe is bounded by ~(245º, 343º) away from the first lobe.



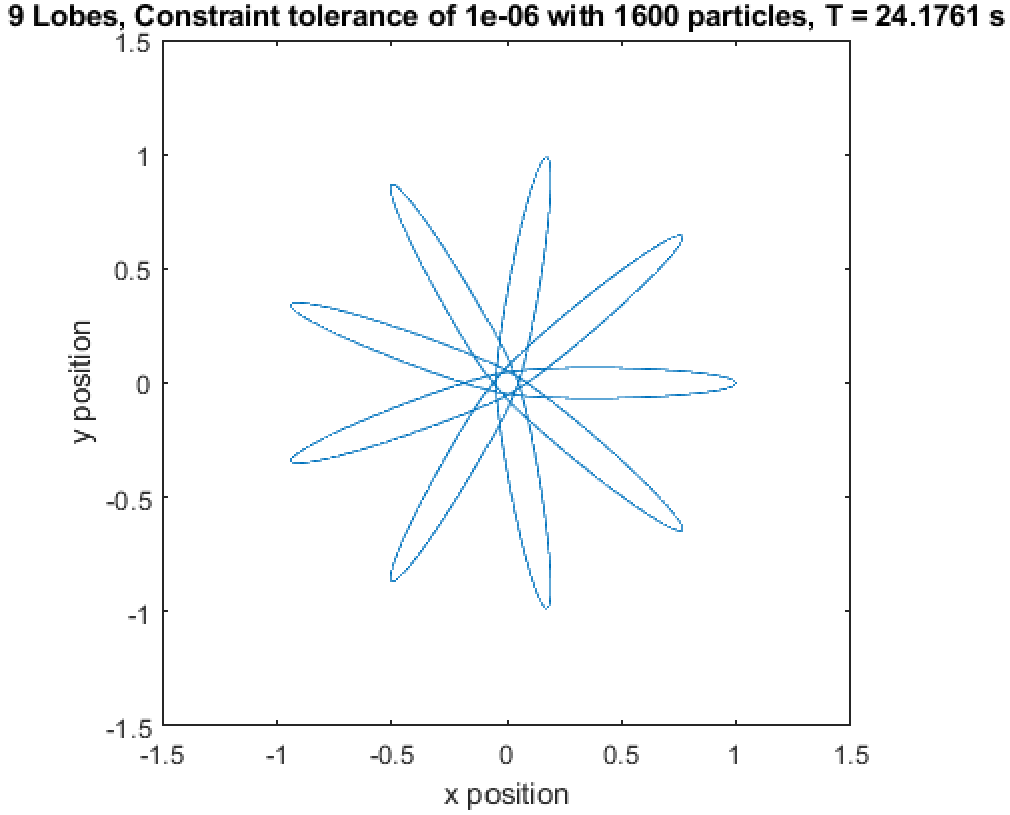


Central Force $F = -r^{-\frac{3}{2}}$

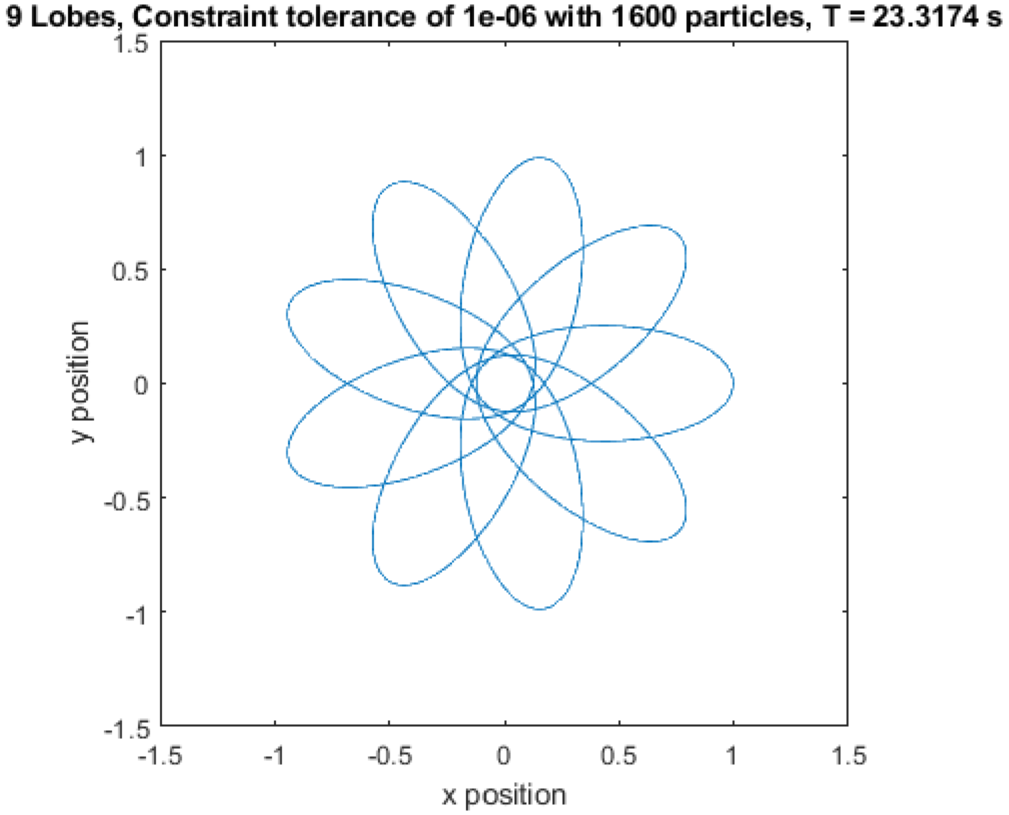




Central Force $F = -r^{-\frac{1}{2}}$



Central Force $F = -r^{-\frac{3}{2}}$



Takeaways

- For $F = r^{-\frac{1}{2}}$, there does not exist a 3,4,6,10,14 lobed solution.
- For some lobed solutions, there exist multiple solutions, although their trajectory do not look identical.
- A particle subject to a central force is bounded by different angles. By looking at $[x\ y\ \dot{x}\ \dot{y}]$ = $[1\ 0\ 0\ 1]$, $v\ \in [0, 1]$, we can capture all the possible lobed solutions for a given central force.
- There appears to always be at least one solution after 14 lobes. (at least until 3000)
- The possible lobed solutions for each central force differs.
- Shooting is better when the guess is very close to the desired solution.
- To help convergence, it is sometimes useful to iterate, stepping up the tolerance.
- Impose constraints as tight as possible; double check that they are reasonable for the desired goal.